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## SIGNIFICANT DIGIT COMPUTATION OF THE ELLIPTICAL COVERAGE FUNCTION

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FOREWORD

The work described in this report was done in the Space and Surface Systems Division with partial support from the Computer and Information Systems Division of the Strategic Systems Department. Its purpose was to develop a new algorithm for the elliptical coverage function, and to design associated Fortran software which is suitable for inclusion in a high quality mathematics and/or statistics subroutine library.

The author is indebted to Alfred H. Morris, Jr. of the Naval Surface Warfare Center (NAVSWC) for suggestions which led to improvement in the algorithms, code, and clarity of the report. He also performed some checking of the software with programs he developed for that purpose.

This document was administratively reviewed by J. L. Sloop, Head, Space and Surface Systems Division.

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## I. INTRODUCTION

The elliptical coverage function  $P(\bar{R}, \bar{\sigma}_x, \bar{\sigma}_y, \bar{h}, \bar{k})$  represents the probability of a shot or event occurring within a circle  $C$  in the  $xy$ -plane with radius  $\bar{R}$  and center  $(\bar{h}, \bar{k})$ , i.e.,

$$C: (x - \bar{h})^2 + (y - \bar{k})^2 = \bar{R}^2.$$

The shot falls under an uncorrelated bivariate normal distribution with mean  $(0,0)$  and standard deviations  $\bar{\sigma}_x, \bar{\sigma}_y$  in the  $x$  and  $y$  directions, respectively. Thus

$$P(\bar{R}, \bar{\sigma}_x, \bar{\sigma}_y, \bar{h}, \bar{k}) = \frac{1}{2\pi\bar{\sigma}_x\bar{\sigma}_y} \int \int_{\text{Area of } C} E[x_1/(\sqrt{2}\bar{\sigma}_x)] E[y_1/(\sqrt{2}\bar{\sigma}_y)] dx_1 dy_1 \quad (1)$$

$$P(\bar{R}, \bar{\sigma}_x, \bar{\sigma}_y, \bar{h}, \bar{k}) = P(\bar{R}, \bar{\sigma}_y, \bar{\sigma}_x, \bar{k}, \bar{h}) = P(\bar{R}, \bar{\sigma}_x, \bar{\sigma}_y, |\bar{h}|, |\bar{k}|), \quad (2)$$

where

$$E(z) \equiv \exp(-z^2). \quad (3)$$

Letting  $x_1 = \sqrt{2}\bar{\sigma}_x x$ , and  $y_1 = \sqrt{2}\bar{\sigma}_y y$ , one obtains

$$P = \frac{1}{\pi} \int_{(\bar{h} - \bar{R})/(\sqrt{2}\bar{\sigma}_x)}^{(\bar{h} + \bar{R})/(\sqrt{2}\bar{\sigma}_x)} E(x) \int_{(\bar{k} - \sqrt{\bar{R}^2 - (\bar{h} - \sqrt{2}\bar{\sigma}_x x)^2})/(\sqrt{2}\bar{\sigma}_y)}^{(\bar{k} + \sqrt{\bar{R}^2 - (\bar{h} - \sqrt{2}\bar{\sigma}_x x)^2})/(\sqrt{2}\bar{\sigma}_y)} E(y) dy dx. \quad (4)$$

It is easily seen from (4) that  $P$  is a function of four independent variables. Therefore, taking (2) into account, we assume hereafter that  $\bar{\sigma}_x \geq \bar{\sigma}_y > 0$ , and that all variables are normalized by  $\bar{\sigma}_x$  with an additional factor of  $1/\sqrt{2}$  for  $\bar{R}, \bar{h}, \bar{k}$ . Hence

$$R = \frac{\bar{R}}{\sqrt{2}\bar{\sigma}_x}, \quad h = \frac{\bar{h}}{\sqrt{2}\bar{\sigma}_x}, \quad k = \frac{\bar{k}}{\sqrt{2}\bar{\sigma}_x}, \quad \sigma_y = \frac{\bar{\sigma}_y}{\bar{\sigma}_x} (\leq 1), \quad \sigma_x = 1. \quad (5)$$

Splitting the  $x$ -integration interval in (4) into  $[h - R, h]$  and  $[h, h + R]$  and using the transformation  $x = h - Ru$  in the first integral and  $x = h + Ru$  in the second integral gives

$$P = \frac{R}{2\sqrt{\pi}} \int_0^1 [E(h - Ru) + E(h + Ru)] \text{aerf}(k_y, R_y \sqrt{1 - u^2}) du, \quad (6)$$

where

$$k_y = k/\sigma_y, \quad R_y = R/\sigma_y, \quad (7)$$

and, for all real  $p$  and  $q$ ,

$$\text{aerf}(p, q) \equiv \text{erf}(p + q) - \text{erf}(p - q) \quad (8)$$

$$\text{erf } p \equiv \frac{2}{\sqrt{\pi}} \int_0^p E(z) dz .$$

In (6), the square root singularity at  $u = 1$  is eliminated by letting

$$t = \sqrt{1-u} .$$

Hence,

$$P = \frac{R}{\sqrt{\pi}} \int_a^b \left\{ E[h - R(1-t^2)] + E[h + R(1-t^2)] \right\} \text{aerf}(k_y, R_y t \sqrt{2-t^2}) t dt \quad (9)$$

where  $a = 0$  and  $b = 1$ .

Relation (9) appears in [2, 3, 4]. In those papers the objective was to give a numerical procedure to compute  $P$  to 6 decimal digit absolute accuracy with the practical limitations:

$$1/15 \leq \sigma_y \leq 15, \quad 0 \leq \sqrt{2} h \leq 600, \quad 0 \leq \sqrt{2} k \leq 600, \quad 10^{-6} \leq P \leq 1 - 10^{-6} . \quad (10)$$

In this report an algorithm is given for computing  $P$  which is based on relative rather than absolute error. A new subroutine, written in Fortran, using this algorithm is contained in the Naval Surface Warfare Center (NAVSWC) Library of Mathematics Subroutines (NSWCLIB), [8]. The new subroutine is called PKILL and it replaces the previous version which was also called PKILL.  $P$  is now computed to at least 6 significant digits provided that

$$10^{-20} \leq P \leq 1 - 10^{-10}, \quad \text{and} \quad \max[h, k_y, 1/(\sqrt{2} \sigma_y)] \leq 5 \sqrt{2/\text{EPS}} ,$$

where EPS denotes the smallest, machine dependent, positive number such that

$$1.0 + \text{EPS} > 1.0 . \quad (11)$$

For the IBM PC double precision arithmetic  $\text{EPS} = 2^{-52} \simeq 2.222 \cdot 10^{-16}$ ; for the CDC 6000 - 7000 series single precision arithmetic  $\text{EPS} = 2^{-47} \simeq 7.105 \cdot 10^{-15}$ . If  $\sigma_y$ ,  $h$ ,  $k$  and  $P$  satisfy (10), then  $P$  is accurate to at least 8 significant digits.

It is worth mentioning that there is much to be gained from the increased robustness and accuracy made available by the new subroutine, PKILL. Indeed, in the future it will be possible to design an inverse procedure by which  $R$  is found as a function of  $P$ ,  $\sigma_y$ ,  $h$ ,  $k$ . Also, since  $P$  is used to compute its 3-dimensional analog [5], increased robustness and relative accuracy will also be achieved for that computation.

## II. DISCUSSION OF THE ALGORITHM

In order to compute  $P$  with relative accuracy using (9), or its equivalent with the order of integration reversed, three main sources for loss of digits are recognized:

- (a) If  $\sigma_y \ll 1$ , inherent error can play a major role when  $R$  is large and is very close to  $k$  or  $\sqrt{h^2 + k^2}$ . When this inherent error dominates the generated error, PKILL does the best it can with no claim on relative accuracy.
- (b) A loss of digits was possible in the past in the computation of  $\text{aerf}(x, y)$  when  $y \ll x$ . A routine has been written, called AERF and contained in NSWCLIB, which gives aerf to 12 significant digits. It is described in Appendix A.
- (c) Truncation error can occur due to using 32 order Gaussian quadrature to numerically evaluate  $P$ .

Extensive testing was carried out to insure the accuracy claimed. This testing, using a Compaq Deskpro 386/20 PC, compared PKILL double precision results, with the results from a double precision adaptive quadrature subroutine, DQAGS, which is contained in NSWCLIB. Also, some additional checks were carried out by Alfred Morris, Jr. (NAVSWC) using an integrator he developed for this purpose.

PKILL initially processes a few tests to check if  $P$  can be set to 1 or 0.  $P$  is set to 0 when these tests indicate that  $P < 10^{-50}$ ; if  $P$  is estimated to be greater than  $1 - 10^{-10}$  then  $P$  is set to 1. These tests are described in [4].

If  $\sigma_y$  is sufficiently small with  $R \geq k$ , then  $P$  can often be evaluated without resorting to numerical quadrature. We reverse the integration in (4) and use (5) to get

$$P = \frac{1}{\pi} \int_{k_y - R_y}^{k_y + R_y} E(y) J(y, \sigma_y) dy, \quad (12)$$

where

$$J(y, \sigma_y) = \frac{\sqrt{\pi}}{2} \text{aerf}\left(h, \sqrt{R^2 - (k - \sigma_y y)^2}\right).$$

For convenience,  $J(y, \sigma_y)$  will also be denoted by  $J$ .

If  $R > k$ , then for sufficiently small  $\sigma_y$ , the integration limits on  $y$  in (12) can be set to  $-\infty$  and  $\infty$ . We proceed by obtaining the Maclaurin expansion of  $J$  in  $\sigma_y$ . The expansion to second order is

$$J \simeq J_0 + J_0' \sigma_y + J_0'' \sigma_y^2 / 2, \quad J_0 \equiv J(y, 0), \quad J_0' \equiv \frac{\partial J}{\partial \sigma_y}(y, 0), \quad (13)$$

where

$$J_0 = \frac{\sqrt{\pi}}{2} \text{aerf}\left(h, \sqrt{R^2 - k^2}\right) \quad (14)$$

$$J_0' = \frac{ky}{\sqrt{R^2 - k^2}} [E(h - \sqrt{R^2 - k^2}) + E(h + \sqrt{R^2 - k^2})] \quad (15)$$

$$J_0'' = \frac{y^2}{\sqrt{R^2 - k^2}} \left\{ \left( -\frac{R^2}{R^2 - k^2} + \frac{2hk^2}{\sqrt{R^2 - k^2}} - 2k^2 \right) E(h - \sqrt{R^2 - k^2}) \right. \\ \left. - \left( \frac{R^2}{R^2 - k^2} + \frac{2hk^2}{\sqrt{R^2 - k^2}} + 2k^2 \right) E(h + \sqrt{R^2 - k^2}) \right\}. \quad (16)$$

Now  $\int_{-\infty}^{\infty} E(y) J_0' dy = 0$  since  $\int_{-\infty}^{\infty} y E(y) dy = 0$ . Therefore for small  $\sigma_y$ ,  $P$  can be approximated

by

$$P \simeq \frac{1}{\pi} \int_{-\infty}^{\infty} E(y) J_0 dy = \frac{1}{\sqrt{\pi}} J_0 = \frac{1}{2} \operatorname{erf}(h, \sqrt{R^2 - k^2}) \quad (17)$$

provided that

$$\sigma_y^2 \frac{1}{\pi} \int_{-\infty}^{\infty} E(y) |J_0''| dy \leq \epsilon \operatorname{erf}(h, \sqrt{R^2 - k^2}). \quad (18)$$

In PKILL, (17) is used when

$$\frac{1}{\sqrt{\pi}} \frac{\sigma_y^2}{R^2 - k^2} \left\{ \left( \frac{R^2}{\sqrt{R^2 - k^2}} + 2k^2 \right) |h - \sqrt{R^2 - k^2}| \right\} \leq \epsilon \operatorname{erf}(h, \sqrt{R^2 - k^2}). \quad (19)$$

The left hand side of (19) is an upper bound for the left hand side of (18) since

$$\int_{-\infty}^{\infty} y^2 E(y) dy = \frac{\sqrt{\pi}}{2}.$$

The quantity  $\epsilon$  is given by

$$\epsilon = \min(6.71\sqrt{\text{EPS}}, 10^{-5}). \quad (20)$$

For the case of sufficiently small  $\sigma_y$  and  $R = k$ , a similar analysis is used. Now

$$J = \operatorname{erf}(h, -z \sqrt{1 - c z^2}) \quad (21)$$

where

$$r = \sqrt{2Ry}, \quad c = y/(2R), \quad z = \sqrt{\sigma_y}.$$

Expanding  $J$  to third order in  $z$  gives

$$J \simeq J_0 + J_0' z + J_0'' z^2/2 + J_0''' z^3/6. \quad (22)$$

But  $J_0 = 0$  and  $J_0'' = 0$ , so that

$$J \simeq J_0' z + J_0''' z^3/6. \quad (23)$$

Then from

$$\frac{\partial J}{\partial z} = \frac{r(1-2cz^2)}{\sqrt{1-cz^2}} \left[ E(h+rz\sqrt{1-cz^2}) + E(h-rz\sqrt{1-cz^2}) \right], \quad (24)$$

we have

$$J_0' = 2rE(h), \quad (25)$$

and from  $\partial^3 J / \partial z^3$  we obtain

$$J_0'''/6 = 2r[-c/2 + r^2(2h^2-1)/3]E(h). \quad (26)$$

Hence

$$P \simeq 2z\sqrt{2R}E(h) \frac{1}{\pi} \int_0^\infty E(y)\sqrt{y} dy \quad (27)$$

or

$$P \simeq \frac{2}{\pi} \sqrt{2R\sigma_y} E(h) \int_0^\infty E(y)\sqrt{y} dy = \frac{1}{\pi} E(h) \Gamma(\frac{3}{4}) \sqrt{2R\sigma_y} \quad (28)$$

provided that

$$2\sigma_y \left| \frac{2}{3}R(2h^2-1) + \frac{1}{4R} \right| \int_0^\infty E(y)y\sqrt{y} dy / \Gamma(\frac{3}{4}) \leq \epsilon, \quad (29)$$

where

$$\int_0^\infty E(y)y\sqrt{y} dy = \Gamma(\frac{5}{4})/2, \quad \Gamma(\frac{5}{4})/\Gamma(\frac{3}{4}) \simeq .73966878.$$

If the approximations for  $P$  do not apply then (9), or the corresponding relation with the order of integration reversed, is evaluated by using a 32 order Gaussian quadrature. The Gaussian abscissas and weights are denoted by  $V(|j|)$  and  $W(|j|)$ , respectively,  $|j| \leq 16$ ,  $j \neq 0$ , [1, p.917]. Thus from (9), with  $b$  and  $a$  denoting the upper and lower limits of integration,  $P$  is approximated by

$$P \sim \frac{b-a}{2} \frac{R}{\sqrt{\pi}} \sum_{\substack{n=1 \\ n \neq 17}}^{33} W(|j|) G_j t_j, \quad j = n-17, \quad (30)$$

where

$$t_j = \text{sign}(j) \frac{b-a}{2} V(|j|) + \frac{b+a}{2}, \quad (31)$$

$$z(t_j) = R_y t_j \sqrt{2-t_j^2},$$

$$G_j = \{E[h-R(1-t_j^2)] + E[h+R(1-t_j^2)]\} \text{aerf}[k_y, z(t_j)].$$

The length of the integration interval,  $b-a$ , can often be reduced because of the spike-like character of the integrand in (9) for large  $h$  and/or  $k$ . In [3, p.378], a parameter  $a$  was introduced which, for a given  $\epsilon$ , led to an effective integration interval  $[a, b] \subseteq [0, 1]$ . That determination of  $a$  was based on absolute accuracy requirements. Here this parameter, which we now call  $A$ , depends on the value of  $P$  because of the more stringent relative accuracy demands. It ranges from 6.5 to 16.

Rough order of magnitude estimates of  $P$  are used to determine  $A$ . These estimates are given by T9, YY, and ZZ below.

$$T9 = \begin{cases} R E(h) \operatorname{aerf}(k_y, R_y) & \sigma_y < .05, \quad k_y > R_y \\ 1 & \text{otherwise} \end{cases}$$

$$YY = \frac{1}{4} \operatorname{aerf}(h, R) \operatorname{aerf}(k_y, R_y) > P \quad (32)$$

ZZ =  $\gamma(c, x)$  (Incomplete gamma function, [6])

$$c = [1 + \sigma_y^2 + 2(h^2 + k^2)]^2 / T$$

$$x = 2R^2 [1 + \sigma_y^2 + 2(h^2 + k^2)] / T$$

$$T = 2[1 + \sigma_y^4 + 4(h^2 + k^2 \sigma_y^2)].$$

The expression ZZ was given by F. Grubbs, [7]. The estimate we choose for P, which was determined by numerical testing, is denoted by  $Z_1$  and is given by

$$Z_1 = \begin{cases} YY & \text{if } YY < 5 \cdot 10^{-15} \\ \min(YY, ZZ) & \text{otherwise.} \end{cases}$$

The procedure for selecting A, using  $Z_1$  and  $T9$ , is given in the Fortran listing of PKILL in Appendix B (see page B-5).

It was shown in [2, 3] that for a given value of A, with  $\beta \equiv A/\sqrt{2}$ , the interval of integration may be reduced according to the following expressions:

$$b = \begin{cases} \sqrt{1 - (h - \beta)/R} & \text{if } 0 < (h - \beta)/R < 1 \\ 1 & \text{if } (h - \beta) \leq 0 \\ 0 & \text{if } (h - \beta)/R \geq 1 \end{cases} \quad (33)$$

$$a = \begin{cases} \sqrt{1 - (h + \beta)/R} & \text{if } (h + \beta)/R < 1 \\ 0 & \text{if } (h + \beta)/R \geq 1. \end{cases} \quad (34)$$

The interval of integration may possibly be reduced further by the following considerations. Let

$$f(t) = \operatorname{aerf}[k_y, z(t)], \quad z(t) = R_y t \sqrt{2 - t^2} \quad (\text{see (9)}).$$

The function  $k_y - z(t)$  decreases from  $k_y$  to  $k_y - R_y$ . We already know that if  $k_y - R_y > \beta$ , then  $f \simeq 0$  since  $6.5/\sqrt{2} \leq \beta \leq 16/\sqrt{2}$ . Hence there may exist a value of  $t$ , say  $\bar{t}$ , such that  $k_y - z(\bar{t}) = \beta$  in which

case the integration in (30) need only be carried out from  $a = \max[\bar{t}, a]$  (from (34)) to  $b$ . The quantity  $\bar{t}$  is found explicitly from  $k_y - z(\bar{t}) = \beta$  and is given by

$$\bar{t} = \alpha / [1 + \sqrt{1 - \alpha^2}]^{1/2}, \quad \text{provided } 0 < \alpha < 1, \quad \text{where } \alpha \equiv (k_y - \beta) / R_y. \quad (35)$$

Now let

$$E = (b - a)/2, \quad D = (b + a)/2, \quad F = 1 - D,$$

where  $E$  and  $D$  appear in (30) and (31) and  $F$  is used in (30) to compute  $1 - t_j^2$  in  $G_j$ . In order to summarize the results for  $E$ ,  $D$ , and  $F$  let

$$H = (h - \beta)/R, \quad Z = (h + \beta)/R, \quad U = \sqrt{1 - \alpha^2}. \quad (36)$$

We note that  $H < 1.0$ , otherwise  $P \simeq 0.0$  as noted in (33).

Then if  $H \leq 0, Z \geq 1$ ,

$$D = (1 + \alpha / \sqrt{1 + U})/2, \quad E = F = U/(4D), \quad \text{if } U < Z,$$

$$E = D = F = 1/2, \quad \text{otherwise.}$$

If  $H \leq 0, Z < 1$ ,

$$D = (1 + \alpha / \sqrt{1 + U})/2, \quad E = F = U/(4D), \quad \text{if } U < Z,$$

$$D = (1 + \sqrt{1 - Z})/2, \quad E = F = Z/(4D), \quad \text{otherwise.}$$

If  $H > 0, Z < 1$ ,

$$D = [\sqrt{1 - H} + \sqrt{1 - U}]/2, \quad E = (U - H)/(4D), \quad 2F = \frac{H}{1 + \sqrt{1 - H}} + \frac{U}{1 + \sqrt{1 - U}}, \quad \text{if } H < U < Z,$$

$$D = [\sqrt{1 - H} + \sqrt{1 - Z}]/2, \quad E = \beta/(2R D), \quad 2F = \frac{H}{1 + \sqrt{1 - H}} + \frac{Z}{1 + \sqrt{1 - Z}}, \quad \text{otherwise.}$$

If  $H > 0, Z \geq 1$ ,

$$D = [\sqrt{1 - H} + \sqrt{1 - U}]/2, \quad E = (U - H)/(4D), \quad 2F = \frac{H}{1 + \sqrt{1 - H}} + \frac{U}{1 + \sqrt{1 - U}}, \quad \text{if } H < U < Z,$$

$$E = D = \frac{1}{2} \sqrt{1 - H}, \quad F = \frac{1}{2} (1 + \frac{H}{1 + \sqrt{1 - H}}), \quad \text{otherwise.}$$

Once  $E$ ,  $D$ , and  $F$  are computed, the order of integration is reversed, by interchanging  $\bar{h}$ ,  $\bar{k}$  and  $\bar{\sigma}_x$ ,  $\bar{\sigma}_y$ , and  $E$ ,  $D$ , and  $F$  are computed again. Comparisons of the two sets of results for  $E$ ,  $D$ , and  $F$  are then used to determine the order of integration to be used in evaluating  $P$ . The details can be seen by looking at the PKILL code in Appendix B (see page B-8).

## III. NUMERICAL RESULTS

The accuracy of the results from PKILL was established by extensive computer testing. An inverse procedure, using PKILL, gave values of  $\bar{R}$  for given values of  $P_0$ ,  $\bar{\sigma}_x$ ,  $\bar{\sigma}_y$ ,  $\bar{h}$ ,  $\bar{k}$ . Then an adaptive integration subroutine, called DQAGS, which is contained in NSWCLIB [8], was used to compute values of  $P$ , denoted by  $P_T$ , for given values of  $P_0$ ,  $\bar{R}$ ,  $\bar{\sigma}_x$ ,  $\bar{\sigma}_y$ ,  $\bar{h}$ ,  $\bar{k}$ . The results were compared with those from PKILL, and the relative error (RERR) was found from  $|P_T - P_K|/P_0$ , where  $P_K$  was obtained from PKILL. A few numerical results are tabulated below.

Case No.	$P_0 = 1 \cdot 10^{-10}$	$\bar{\sigma}_x = 1.0$	$\bar{R}$	$\bar{h}$	$\bar{k}$	$\bar{\sigma}_y$	RERR
1.	1.00000000003092E-5		0.00	0.00	.500	0.0E-00	
2.	4.47213595556272E-6		0.00	0.00	.100	9.0E-16	
3.	4.41421356414086E-6		0.00	0.00	.010	2.6E-16	
4.	2.71828182707694E-5		0.00	1.00	.500	2.6E-16	
5.	4.04044149100285E-1		0.00	1.00	.100	2.6E-11	
6.	9.41563620089423E-1		0.00	1.00	.010	1.7E-14	
7.	1.28402541674078E-5		1.00	0.00	.500	1.2E-14	
8.	5.74233623428373E-6		1.00	0.00	.100	7.0E-15	
9.	1.81588616245426E-6		1.00	0.00	.010	5.3E-15	
10.	1.93372255656656E0		1.00	5.00	.500	4.5E-15	
11.	4.39270699558773E0		1.00	5.00	.100	1.1E-09	
12.	4.94102329874250E0		1.00	5.00	.010	2.4E-14	
13.	1.40747750223962E-2		5.00	1.00	.500	3.9E-16	
14.	6.43254811832100E-1		5.00	1.00	.100	1.3E-16	
15.	9.67161512902738E-1		5.00	1.00	.010	5.2E-16	
16.	7.50347341230299E0		5.00	10.0	.500	2.6E-10	
17.	9.56989445416763E0		5.00	10.0	.100	2.0E-12	
18.	9.96348038065810E0		5.00	10.0	.010	6.5E-16	
19.	5.61991593272086E0		10.0	5.00	.500	2.8E-12	
20.	6.15159059893341E0		10.0	5.00	.100	2.7E-09	
21.	6.18350703594593E0		10.0	5.00	.010	1.1E-08	
22.	5.06546585192287E2		100.	500.	.500	5.8E-09	
23.	5.08536986866509E2		100.	500.	.100	1.4E-09	
24.	5.08690971362607E2		100.	500.	.010	4.7E-08	

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$$P_0 = 1 \cdot 10^{-10} \quad \bar{\sigma}_x = 1.0$$

Case No.	$\bar{R}$	$\bar{h}$	$\bar{k}$	$\bar{\sigma}_y$	RERR
25.	5.03634144947275E2	500.	100.	.500	8.4E-08
26.	5.03664419834487E2	500.	100.	.100	6.5E-08
27.	5.03665675807472E2	500.	100.	.010	8.9E-08
28.	9.99997319325154E5	1000.	1E6	.500	2.0E-08
29.	9.99999863834799E5	1000.	1E6	.100	2.2E-08
30.	1.00000043606992E6	1000.	1E6	.010	2.1E-07
31.	9.99994138661486E5	1E6	1000.	.500	2.1E-08
32.	9.99994138662153E5	1E6	1000.	.100	1.7E-07
33.	9.99994138662153E5	1E6	1000.	.010	1.5E-09

$$P_0 = .0050 \quad \bar{\sigma}_x = 1.0$$

Case No.	$\bar{R}$	$\bar{h}$	$\bar{k}$	$\bar{\sigma}_y$	RERR
34.	7.08214973500709E-2	0.00	0.00	.500	8.7E-16
35.	3.18243343370606E-2	0.00	0.00	.100	1.4E-15
36.	1.06881353713506E-2	0.00	0.00	.010	6.9E-16
37.	1.87773801239816E-1	0.00	1.00	.500	5.2E-16
38.	8.10127444936150E-1	0.00	1.00	.100	8.0E-13
39.	9.85554779402774E-1	0.00	1.00	.010	6.6E-11
40.	9.09820386010898E-2	1.00	0.00	.500	1.7E-15
41.	4.10294072289614E-2	1.00	0.00	.100	5.2E-16
42.	1.43927001021222E-2	1.00	0.00	.010	1.0E-15
43.	3.84954308155358E0	1.00	5.00	.500	1.0E-13
44.	4.79375448729158E0	1.00	5.00	.100	1.5E-12
45.	4.98408394392712E0	1.00	5.00	.010	9.1E-11
46.	2.62868451641037E0	5.00	1.00	.500	1.7E-16
47.	2.62253474480775E0	5.00	1.00	.100	2.4E-12
48.	2.62232994242921E0	5.00	1.00	.010	2.6E-12
49.	9.64438962680617E0	5.00	10.0	.500	5.8E-13
50.	1.02483865120650E1	5.00	10.0	.100	2.4E-12
51.	1.02892071201973E1	5.00	10.0	.010	3.2E-12

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 $P_0 = .0050$        $\bar{\sigma}_x = 1.0$ 

Case No.	$\bar{R}$	$\bar{h}$	$\bar{k}$	$\bar{\sigma}_y$	RERR
52.	8.85881461639006E0	10.0	5.00	.500	1.3E-11
53.	8.94686556206516E0	10.0	5.00	.100	4.3E-12
54.	8.95084289534896E0	10.0	5.00	.010	9.6E-12
55.	5.08543123356466E2	100	500	.500	2.6E-10
56.	5.09342260944039E2	100	500	.100	1.9E-10
57.	5.09402405988742E2	100	500	.010	3.6E-12
58.	5.07363938745357E2	500	100	.500	7.8E-12
59.	5.07375894420008E2	500	100	.100	2.8E-12
60.	5.07376389398212E2	500	100	.010	2.3E-12
61.	9.99999212083791E5	1000	1E6	.500	2.2E-11
62.	1.00000024240470E6	1000	1E6	.100	5.4E-10
63.	1.00000047411365E6	1000	1E6	.010	1.6E-07
64.	9.99997924171662E5	1E6	1000	.500	8.2E-11
65.	9.99997924171859E5	1E6	1000	.100	2.2E-08
66.	9.99997924171859E5	1E6	1000	.010	2.7E-11

 $P_0 = .500$        $\bar{\sigma}_x = 1.0$ 

Case No.	$\bar{R}$	$\bar{h}$	$\bar{k}$	$\bar{\sigma}_y$	RERR
67.	8.70417428244252E-1	0.00	0.00	.500	0.0E-00
68.	6.81985088272086E-1	0.00	0.00	.100	6.7E-16
69.	6.74563888092905E-1	0.00	0.00	.010	2.5E-12
70.	1.35499509964184E0	0.00	1.00	.500	1.3E-15
71.	1.22603637241642E0	0.00	1.00	.100	4.3E-15
72.	1.20638167322879E0	0.00	1.00	.010	2.5E-12
73.	1.18221140061131E0	1.00	0.00	.500	6.7E-16
74.	1.05532095681235E0	1.00	0.00	.100	1.9E-12
75.	1.05059188896165E0	1.00	0.00	.010	2.6E-12

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 $P_0 = .500$      $\bar{\sigma}_x = 1.0$ 

Case No.	$\bar{R}$	$\bar{h}$	$\bar{k}$	$\bar{\sigma}_y$	RERR
76.	5.18608676231112E0	1.00	5.00	.500	8.9E-16
77.	5.13530939278725E0	1.00	5.00	.100	2.9E-15
78.	5.10946764748624E0	1.00	5.00	.010	2.2E-12
79.	5.12431831343601E0	5.00	1.00	.500	1.0E-12
80.	5.10003905042201E0	5.00	1.00	.100	7.7E-11
81.	5.09902971170382E0	5.00	1.00	.010	8.0E-11
82.	1.12084500495804E1	5.00	10.0	.500	4.2E-15
83.	1.11824977123348E1	5.00	10.0	.100	2.4E-12
84.	1.11803622401548E1	5.00	10.0	.010	2.6E-12
85.	1.11935071779149E1	10.0	5.00	.500	6.9E-11
86.	1.11808975580793E1	10.0	5.00	.100	7.9E-11
87.	1.11803454776247E1	10.0	5.00	.010	8.0E-11
88.	5.09902830500163E2	100.	500.	.500	8.0E-11
89.	5.09902155326263E2	100.	500.	.100	2.4E-12
90.	5.09901953902433E2	100.	500.	.010	3.4E-12
91.	5.09902203786139E2	500.	100.	.500	8.0E-11
92.	5.09901961553339E2	500.	100.	.100	8.0E-11
93.	5.09901951359279E2	500.	100.	.010	8.3E-08
94.	1.00000050000037E6	1000	1E6	.500	1.8E-10
95.	1.00000050000037E6	1000	1E6	.10	4.1E-10
96.	1.00000050000037E6	1000	1E6	.010	1.3E-10
97.	1.00000049999987E6	1E6	1000	.500	1.0E-07
98.	1.00000049999987E6	1E6	1000	.100	4.0E-09
99.	1.00000049999987E6	1E6	1000	.010	1.2E-10

 $P_0 = .999$      $\bar{\sigma}_x = 1.0$ 

Case No.	$\bar{R}$	$\bar{h}$	$\bar{k}$	$\bar{\sigma}_y$	RERR
100.	3.33463215647498E0	0.00	0.00	.500	2.3E-11
101.	3.29205427406123E0	0.00	0.00	.100	8.0E-11
102.	3.29052673159727E0	0.00	0.00	.010	6.7E-16

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 $P_0 = .999$        $\bar{\sigma}_x = 1.0$ 

Case No.	$\bar{R}$	$\bar{h}$	$\bar{k}$	$\bar{\sigma}_y$	RERR
103.	3.53577235751072E0	0.00	1.00	.500	8.9E-16
104.	3.44218941775034E0	0.00	1.00	.100	8.0E-11
105.	3.43912287816784E0	0.00	1.00	.010	1.1E-07
106.	4.12431141206616E0	1.00	0.00	.500	7.4E-11
107.	4.09151266085973E0	1.00	0.00	.100	8.0E-11
108.	4.09028540157306E0	1.00	0.00	.010	1.1E-16
109.	6.93719921979876E0	1.00	5.00	.500	1.2E-15
110.	6.47656370770719E0	1.00	5.00	.100	8.0E-11
111.	6.46007535448830E0	1.00	5.00	.010	8.0E-11
112.	8.17462362868007E0	5.00	1.00	.500	3.9E-11
113.	8.15266008069658E0	5.00	1.00	.100	8.0E-11
114.	8.15180095237486E0	5.00	1.00	.010	2.9E-08
115.	1.32590217041336E1	5.00	10.0	.500	2.2E-14
116.	1.28786495588648E1	5.00	10.0	.100	8.0E-11
117.	1.28629674529177E1	5.00	10.0	.010	8.0E-11
118.	1.40783138161510E1	10.0	5.00	.500	7.7E-11
119.	1.40151628274090E1	10.0	5.00	.100	8.0E-11
120.	1.40126436416327E1	10.0	5.00	.010	9.1E-08
121.	5.11535358789872E2	100.	500.	.500	6.9E-11
122.	5.10586815818419E2	100.	500.	.100	6.3E-11
123.	5.10517724887103E2	100.	500.	.010	8.3E-11
124.	5.12947737652262E2	500.	100.	.500	8.0E-11
125.	5.12933140365073E2	500.	100.	.100	8.0E-11
126.	5.12932531471364E2	500.	100.	.010	2.1E-08
127.	1.00000204511886E6	1000.	1E6	.500	7.8E-11
128.	1.00000080903890E6	1000.	1E6	.100	7.5E-11
129.	1.00000053105685E6	1000.	1E6	.010	2.1E-10
130.	1.000000359023064E6	1E6	1000.	.500	1.7E-09
131.	1.000000359023064E6	1E6	1000.	.100	6.9E-11
132.	1.000000359023064E6	1E6	1000.	.010	2.4E-13

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APPENDIX A

ALGORITHM AND FORTRAN LISTING FOR AERF

## ALGORITHM AND FORTRAN LISTING FOR AERF

In this appendix the algorithm used in PKILL for the computation of the function aerf is given where, for all real  $x$  and  $h$ ,

$$f(x, h) = aerf(x, h) = \operatorname{erf}(x + h) - \operatorname{erf}(x - h) \quad A(1)$$

or

$$f(x, h) = aerf(x, h) = \operatorname{erfc}(x - h) - \operatorname{erfc}(x + h) \quad A(2)$$

$$\operatorname{erf} y = \frac{2}{\sqrt{\pi}} \int_0^y E(z) dz, \quad E(z) = \exp(-z^2)$$

$$\operatorname{erfc} y = 1 - \operatorname{erf} y = \frac{2}{\sqrt{\pi}} \int_y^{\infty} E(z) dz.$$

(As an aside, note that  $\frac{2}{\sqrt{\pi}} \int_a^b E(z) dz = aerf(x, h)$  where  $x = \frac{b+a}{2}$  and  $h = \frac{b-a}{2}$ .)

It is sufficient hereafter to consider  $x > 0$  and  $h > 0$ , since

$$f(-x, h) = f(x, h), \quad f(x, -h) = -f(x, h) \quad \text{and} \quad f(0, h) = 2 \operatorname{erf} h, \quad f(x, 0) = 0.$$

Also, if  $(x - h) > 9.0$ , then  $f(x, h)$  can be set to zero if

$$(x - h)^2 + 2.76959 \geq -\operatorname{DXPAR}(1). \quad A(3)$$

Here  $\operatorname{DXPAR}(1)$  is the smallest negative number such that

$$\exp[\operatorname{DXPAR}(1)] > 0.$$

For the IBM PC double precision arithmetic  $\operatorname{DXPAR}(1) \approx -708.389$ . Also, A(3) follows from

$$f(x, h) < \frac{2}{\sqrt{\pi}} \int_{x-h}^{x+h} E(z) \frac{z}{x-h} dz < \frac{E(x-h)}{\sqrt{\pi}(x-h)} < \exp[-(x-h)^2 - \ln(9\sqrt{\pi})].$$

For sufficiently small  $h$  and/or  $x$  some approximations are used that improve efficiency.

Consider

$$\operatorname{erf}(x+h) = \operatorname{erf}(x) + \frac{2}{\sqrt{\pi}} \int_x^{x+h} E(t) dt,$$

and let  $u = (t-x)/h$  to obtain

$$\operatorname{erf}(x+h) = \operatorname{erf}(x) + \frac{2h}{\sqrt{\pi}} \int_0^1 E(x+hu) du. \quad A(4)$$

Similarly,

$$\operatorname{erf}(x-h) = \operatorname{erf}(x) - \frac{2h}{\sqrt{\pi}} \int_0^1 E(x-hu) du,$$

and

$$f(x, h) = \frac{2h}{\sqrt{\pi}} \int_0^1 [E(x-hu) + E(x+hu)] du, \quad A(5)$$

or

$$f(x, h) = \frac{4h}{\sqrt{\pi}} E(x) \int_0^1 E(hu) \cosh(2xhu) du. \quad A(6)$$

If  $h$  and  $x$  are both sufficiently small, then the exponentials in A(5) can be replaced by the first few terms of their Maclaurin expansions, i.e.,

$$E(x-h) + E(x+h) \simeq 1 - (x-h)^2 + \frac{1}{2}(x-h)^4 + 1 - (x+h)^2 + \frac{1}{2}(x+h)^4.$$

Integration yields

$$f(x, h) \simeq \frac{4h}{\sqrt{\pi}} \left[ (1 - x^2 - \frac{1}{3}h^2) + \frac{1}{2}(x^4 + 2x^2h^2 + \frac{1}{5}h^4) \right].$$

Thus

$$f(x, h) \simeq \frac{4h}{\sqrt{\pi}} (1 - x^2 - \frac{1}{3}h^2), \quad A(7)$$

provided

$$\frac{1}{2}(T^4 + 2T^4 + \frac{1}{5}T^4) = 1.60T^4 \leq \text{EPS},$$

where  $T = \max(x, h)$ , and EPS is given by (11).

Another approximation that is used can be obtained from (A6) by using the fact that if

$$(2hx)^2 \leq 2 \text{EPS}, \quad \text{then} \quad \cosh 2hxu \sim 1.0,$$

and

$$f(x, h) = 2 E(x) \operatorname{erf} h \quad \text{if} \quad hx \leq \sqrt{\text{EPS}/2}. \quad A(8)$$

In addition, if

$$h^2 \leq 3 \text{EPS}, \quad \text{then} \quad \operatorname{erf} h \sim h,$$

and

$$f(x, h) \sim \frac{4h}{\sqrt{\pi}} E(x), \quad \text{if} \quad hx \leq \sqrt{\text{EPS}/2} \quad \text{and} \quad h \leq \sqrt{3 \text{EPS}}. \quad A(9)$$

The final approximation used is given by

$$f(x, h) \sim \operatorname{erfc}(x-h), \quad \text{if} \quad x+h \geq 5.736, \quad \text{with} \quad h \geq x, \quad \text{or if} \quad \exp(-4hx) \leq \text{EPS}. \quad A(10)$$

In the first case  $\operatorname{erfc}(x-h) \geq 1$ , and  $\operatorname{erfc}(x+h) < 4.99 \cdot 10^{-16}$ . In the second case, we have by using (A2) that

$$f(x, h) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} E(u-h) [1 - \exp(-4hu)] du.$$

Hence, since

$$4hu \geq 4hx \geq -\log(\text{EPS}), \quad x \leq u < \infty, \quad A(11)$$

the approximation given by (A10) holds. A subprogram DESPLN, written by A. Morris and contained in NSWCLIB, supplies  $\log(\text{EPS})$ , where EPS is given in (11).

In case none of the approximations above are applicable for evaluating aerf, then two other procedures are used which follow. One is used if  $x \leq 0.4$  and the other is used when  $x > 0.4$ .

If  $x \leq 0.4$

$$f(x, h) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{[(x+h)^{2n+1} - (x-h)^{2n+1}]}{2n+1}. \quad A(12)$$

This result follows directly by expanding  $(2/\sqrt{\pi})E(z)$  into its Maclaurin series and integrating the series from  $x-h$  to  $x+h$ .

For the computation of  $f(x, h)$  in PKILL let

$$S_n \equiv \frac{u^n - v^n}{u - v} = u S_{n-1} + v^{n-1}, \quad n \geq 1, \quad S_1 = 1. \quad A(13)$$

Then

$$f(x, h) = \frac{4h}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (2n+1)} S_{2n+1}, \quad A(14)$$

where, from (A12) and (A13),

$$\begin{aligned} S_{2n+1} &= (x+h) S_{2n} + (x-h)^{2n} \\ &= (x+h)[(x+h) S_{2n-1} + (x-h)^{2n-1}] + (x-h)^{2n} \\ &= (x+h)^2 S_{2n-1} + 2x(x-h)^{2n-1}, \quad n \geq 1. \end{aligned} \quad A(15)$$

If  $x > 0.40$

$$f(x, h) = \frac{4h}{\sqrt{\pi}} E(x/\sqrt{2}) \sum_{n=0}^{\infty} E(x/\sqrt{2}) \frac{H_{2n}(x) h^{2n}}{(2n+1)!}, \quad A(16)$$

where  $H_n$  denotes the Hermite polynomial of degree  $n$  [1, pp. 781-802]. This series for aerf follows by obtaining the Maclaurin expansions of  $\text{erf}(x+h)$  and  $\text{erf}(x-h)$  and subtracting the resulting series. The expansions for  $\text{erf}(x \pm h)$  are derived by using the following results:

$$\frac{d}{dz} [\text{erf} z] = \frac{2}{\sqrt{\pi}} E(z) \quad A(17.1)$$

$$\frac{d^n}{dz^n} [E(z)] = (-1)^n E(z) H_n(z) \quad A(17.2)$$

$$\frac{d^{n+1}}{dz^{n+1}} [\text{erf} z] = (-1)^n \frac{2}{\sqrt{\pi}} E(z) H_n(z) \quad A(17.3)$$

$$H_{n+1}(z) = 2z H_n(z) - 2n H_{n-1}(z), \quad H_0 = 1, \quad H_1 = 2z, \quad A(17.4)$$

and

$$\text{erf}(x+h) = \text{erf}(x) + \frac{2h}{\sqrt{\pi}} E(x) \sum_{n=0}^{\infty} \frac{H_n(x) (-h)^n}{(n+1)!}. \quad A(18)$$

For the computation of (A16) in PKILL, we rewrite (A16) in the form

$$f(x, h) = \frac{4h}{\sqrt{\pi}} E(x/\sqrt{2}) \sum_{n=0}^{\infty} \frac{J_{2n}}{2n+1}, \quad A(19)$$

where

$$J_n \equiv E(x/\sqrt{2}) \frac{H_n(x) h^n}{n!}, \quad n \geq 0. \quad A(20)$$

In addition,

$$\begin{aligned} J_{2n} &= [2h x J_{2n-1} - 2h^2 J_{2n-2}]/2n, \quad n \geq 1, \quad J_0 = E(x/\sqrt{2}), \\ J_{2n-1} &= [2h x J_{2n-2} - 2h^2 J_{2n-3}]/(2n-1), \quad n \geq 2, \quad J_1 = 2h x E(x/\sqrt{2}). \end{aligned} \quad A(21)$$

Since

$$H_{2n}(x) < \exp(x^2/2) 2^{2n+1} n! , \quad [1, p. 787] \quad A(22)$$

use of Stirling's formula for the factorial gives an estimate for  $J_{2n}$ , i.e.,

$$J_{2n} \sim \frac{2 \sqrt{\pi n} h^{2n}}{n!} . \quad A(23)$$

Now since A(19) is only used when  $4hx < -\log(\text{EPS})$  and  $x > 3h$  (see A(11) and page A-5), it follows even for a machine with a 19 digit mantissa that  $h^2 < 43.75/12 \simeq 3.6458$ . Hence, using A(23), no more than 28 terms of A(19) would be needed to maintain a relative error less than  $10^{-12}$  in  $f(x, h)$ .

```

DOUBLE PRECISION FUNCTION AERF (X,H)
C-----
C      COMPUTATION OF ERF(X + H) - ERF(X - H)
C-----
      DOUBLE PRECISION ERF ,ERFC ,DEPSLN ,DPMPAR ,DXPARG
      DOUBLE PRECISION AH ,AX ,A2 ,C ,C1 ,DN2
      DOUBLE PRECISION E ,EPS ,H ,HF ,HG ,H2
      DOUBLE PRECISION H3 ,S ,ST ,T ,U ,V ,X
      DOUBLE PRECISION XMH ,XPH ,X2 ,Z
C-----
C      C = 2/DSQRT(PI),      C1 = ln (9*DSQRT(PI))
C-----
C      DATA C /1.12837916709551257D0/,  C1/2.76959D0/
C-----
C      EPS IS A MACHINE DEPENDENT CONSTANT. EPS IS THE SMALLEST
C      DOUBLE PRECISION FLOATING POINT NUMBER FOR WHICH
C      1.0 + EPS .GT. 1.0.
C-----
C      EPS = DPMPAR(1)
C-----
      AERF = 0.D0
      IF (H .EQ. 0.D0) RETURN
C
      AH = DABS(H)
      AX = DABS(X)
      XPH = AX + AH
      XMH = AX - AH
C
      T = DMAX1(AH, AX)
      T = T*T
      IF ( 1.6*T*T .LT. EPS) GOTO 140
      IF ((AH*AX)**2 .LE. EPS/2) GOTO 150
      IF (AX .LE. AH) GOTO 100
C
      IF (XMH .LT. 9D0) GOTO 5
          IF (XMH*XMH + C1 .GT. -DXPARG(1) ) RETURN
      5  IF (4*AH*AX .GT. -DEPSLN(0)) GOTO 120
C
      IF (AX .GT. 3D0*AH) GOTO 10
      IF (XPH .LT. 1.D0) GOTO 110
      GOTO 130

```

```

C-----
C      FOR (AX LESS THAN OR EQUAL TO .40)
C-----
10  E = DMAX1(1.D-15, EPS)
    IF (AX .GT. 0.40D0) GOTO 30
    H2 = XPH*XPH
    A2 = XMH*XMH
    X2 = AX + AX
    ST = 1.D0
    HF = XMH
    N = 0
    N1 = 1
    DN2 = 1.D0
    S = 0.D0
20      N = N + 1
        N1 = N1 + 2
        DN2 = -DN2/N
        ST = H2*ST + X2*HF
        HF = A2*HF
        T = ST*DN2/N1
        S = S + T
        IF (DABS(T) .GT. E*DABS(S)) GO TO 20
        S = 0.5D0 + (S + 0.5D0)
        AERF = 2*C*AH*S
        GO TO 45
C-----
C      FOR (AX GREATER THAN .40)
C-----
      N = 1
      J = 0
      H2 = 0.D0
      Z = DEXP(-AX*AX/2)
      U = 2*AH*C*Z
      H3 = Z
      V = 2*H*H
      HF = 2*AX*AH
      S = 0.D0
35      H2 = (HF*H3 - V*H2)/N
        N = N + 1
        H3 = (HF*H2 - V*H3)/N
        N = N + 1
        HG = H3/N
        S = S + HG
        IF (DABS(HG) .GT. E*DABS(S)) GO TO 35
        IF (J .NE. 0) GO TO 40
        J = 1
        GO TO 35
40  AERF = U*(S + Z)
45  IF (H .LT. 0.D0) AERF = -AERF
    RETURN

```

```
C-----  
C   SPECIAL CASES  
C-----  
100 IF (XPH .LT. 5.736D0) GOTO 110  
    IF (XMH .GT. -5.6D0) GOTO 120  
    AERF = SIGN(2.D0, H)  
    RETURN  
C  
110 AERF = ERF(XPH) - ERF(XMH)  
    IF (H .LT. 0.D0) AERF = -AERF  
    RETURN  
C  
120 AERF = ERFC(XMH)  
    IF (H .LT. 0.D0) AERF = -AERF  
    RETURN  
C  
130 AERF = ERFC(XMH) - ERFC(XPH)  
    IF (H .LT. 0.D0) AERF = -AERF  
    RETURN  
C  
140 AERF = 2*C*H* (.5D0 + (.5D0 - (X*X + H*H/3)))  
    RETURN  
C  
C           THE VALUE IS 2*DEXP(-X*X)*ERF(H)  
C  
150 T = 2.D0  
    X2 = X*X  
    IF (X2 .GE. EPS) T = 2*DEXP(-X*X)  
    IF (H*H .GE. 3*EPS) GOTO 160  
        AERF = C*H*T  
        RETURN  
160 AERF = T*ERF(H)  
    RETURN  
END
```

NAVSWC TR 90-513

APPENDIX B

FORTRAN LISTING FOR PKILL

## SUBROUTINE PKILL (R0,SX,SY,H,K,P)

C-----  
 C PKILL GIVES THE ELLIPTICAL COVERAGE FUNCTION, WHERE P DE-  
 C NOTES THE PROBABILITY OF A SHOT, NORMALLY DISTRIBUTED WITH  
 C MEAN (0,0) AND STANDARD DEVIATIONS SX, SY IN THE X,Y  
 C DIRECTIONS, RESPECTIVELY, FALLING IN A CIRCLE IN THE XY-  
 C PLANE OF RADIUS R0 AND CENTERED AT (H, K). THE INPUT IS R0,  
 C SX, SY, H, K. THE OUTPUT IS P.  
 C PKILL GIVES AT LEAST 6-SIGNIFICANT DIGIT ACCURACY FOR P PRO-  
 C VIDED MAX(H/S,K/S,MAX(SX,SY)/S) .LE. 10/DSQRT(DPMPAR(1)), WHERE  
 C S = MIN(SX,SY). IF P .LT. 1.E-20 ACCURACY MAY BE LESS THAN 6 SD.  
 C IF SX OR SY .LE. 0, THEN P SET TO (-1E10).  
 C-----

C REFERENCES...  
 C NWL REPORT N0.1710, AUG.1960. MATH OF COMP. OCT. 1961,  
 C PP. 375, 382. NSWC REPORT N0.83-13, NOVEMBER, 1982.  
 C-----

C DOUBLE PRECISION K, K8, V(16), W(16)  
 C-----

C V(\*), W(\*)-- GAUSSIAN ABSCISSAS AND WEIGHTS OF ORDER 32, ON (-1,1).  
 C-----

EXTERNAL AERF ,DPMPAR ,DXPARG ,DEPSLN ,ERF ,ERFC ,GRATIO  
 DOUBLE PRECISION A AERF ,A1 ,A2 ,A3 ,A4  
 DOUBLE PRECISION C ,C8 ,D ,DEPSLN ,DM ,DPMPAR ,DXPARG  
 DOUBLE PRECISION D1 ,D2 ,EPS ,ERF ,ERFC ,E3  
 DOUBLE PRECISION E4 ,F ,G2 ,G3 ,H ,H1  
 DOUBLE PRECISION H2 ,H3 ,H5 ,H6 ,H8 ,P  
 DOUBLE PRECISION Q ,Q1 ,R ,RPI ,RT2 ,R0  
 DOUBLE PRECISION R8 ,SA ,SEPS1 ,SQEPS ,SX ,SY  
 DOUBLE PRECISION S0 ,S1 ,S2 ,S9 ,T ,T1  
 DOUBLE PRECISION T2 ,T3 ,T4 ,T5 ,T9 ,U  
 DOUBLE PRECISION X1 ,X5 ,YY ,Z ,ZA ,Z1  
 DOUBLE PRECISION Z2 ,Z3 ,Z8 ,Z9  
 DATA V(1) /.4830766568773832D-01/, V(2) /.1444719615827965D+00/,  
 \* V(3) /.2392873622521371D+00/, V(4) /.3318686022821276D+00/,  
 \* V(5) /.4213512761306353D+00/, V(6) /.5068999089322294D+00/,  
 \* V(7) /.5877157572407623D+00/, V(8) /.6630442669302152D+00/,  
 \* V(9) /.7321821187402897D+00/, V(10) /.7944837959679424D+00/,  
 \* V(11) /.8493676137325700D+00/, V(12) /.8963211557660521D+00/,  
 \* V(13) /.9349060759377397D+00/, V(14) /.9647622555875064D+00/,  
 \* V(15) /.9856115115452683D+00/, V(16) /.9972638618494816D+00/  
 DATA W(1) /.9654008851472780D-01/, W(2) /.9563872007927486D-01/,  
 \* W(3) /.9384439908080457D-01/, W(4) /.9117387869576388D-01/,  
 \* W(5) /.8765209300440381D-01/, W(6) /.8331192422694676D-01/,  
 \* W(7) /.7819389578707031D-01/, W(8) /.7234579410884851D-01/,  
 \* W(9) /.6582222277636185D-01/, W(10) /.5868409347853555D-01/,  
 \* W(11) /.5099805926237618D-01/, W(12) /.4283589802222668D-01/,  
 \* W(13) /.3427386291302143D-01/, W(14) /.2539206530926206D-01/,  
 \* W(15) /.1627439473090567D-01/, W(16) /.7018610009470097D-02/

```

C-----
C   C = (2**.25)*GAMMA(3/4)/PI
C   RPI = 1/DSQRT(PI)
C   RT2 = DSQRT(2)
C-----
C   DATA A3 /20.0D0/
C   DATA C /.46386480428950042D0/
C   DATA RT2 /1.414213562373095D0/
C   DATA RPI /.5641895835477563D0/
C
C   P = 0.0D0
C   IF (SX .GT. 0.0D0 .AND. SY .GT. 0.0D0) GOTO 5
C   P = -1.D10
C   RETURN
C   5 J = 0
C   EPS = DPMPAR(1)
C   SQEPS = DSQRT(EPS)
C   A = 6.5D0
C-----
C   TEST NO.1 (REPORT 83-13).
C-----
C   Z3 = DMIN1(DSQRT(DSQRT(DPMPAR(2))), 1.D-30)*SX*SY
C   IF (R0*R0 .LE. Z3) RETURN
C
C   H2 = H*H + K*K
C   H8 = DABS(H)
C   K8 = DABS(K)
C   DM = DMAX1(SX,SY)
C-----
C   TEST NO.2 (REPORT 83-13).
C-----
C   IF ((R0 - H8 + A3*SX) .LE. 0.0D0) RETURN
C   IF ((R0 - K8 + A3*SY) .LE. 0.0D0) RETURN
C-----
C   TEST NO.3 (REFORT 83-13).
C-----
C   C8 = -DEPSLN(0)
C   DEXP(-10.3026) = 3.35E(-5)
C   C8 = -DLOG(EPS)
C
C   A4 = DMAX1(C8 - 1.74249D1, 10.3026D0)
C   A4 = DSQRT(A4 + A4)
C   H3 = (R0 - H)*(R0 + H)
C   H5 = (R0 - K)*(R0 + K)
C   T = R0 - A4*DM
C   IF (T .LT. 0.0D0) GOTO 25
C   IF (T*T .LT. H2) GOTO 25
C   IF (R0 .LT. 1/SQEPS) GOTO 20
C   T = H3 -2.0D0*R0*A4*DM + (A4*DM)**2 - K*K

```

```

IF (DABS(H3) .LE. DABS(H5)) GOTO 10
T = H5 -2.0D0*R0*A4*DM + (A4*DM)**2 - H*H
10 IF (T .LT. 0.0D0) GOTO 25
20 P = 1.0D0
RETURN
C-----
C TEST NO.4 (REPORT 83-13).
C-----
25 S0 = DSQRT(H2)
G2 = 0.D0
IF (S0 .LE. R0) GOTO 30
D = ((S0 - R0)/DM)**2
IF (R0*R0*DEXP(-0.5D0*D) .LE. Z3) RETURN
C-----
C SX - SY .LT. EPS
C-----
30 IF (DABS(SX - SY) .GT. 20.0D0*DM*EPS) GOTO 35
H8 = S0
K8 = 0.0D0
IF ((R0 - H8 + A3*DM) .LE. 0.0D0) RETURN
IF (R0 .LT. DM/SQEPS) GOTO 35
J = 1
G2 = (K*K - H3)/((R0 + H8)*DM)
IF (DABS(H3) .LE. DABS(H5)) GOTO 35
G2 = (H*H - H5)/((R0 + H8)*DM)
C-----
C NORMALIZE DMAX1(SX,SY) = 1.
C-----
35 R = R0/DM
S1 = SX/DM
S2 = SY/DM
H8 = H8/DM
K8 = K8/DM
H2 = H2/(DM*DM)
C-----
C S1 = 1 .GE. S2
C-----
IF (S1 .GE. S2) GOTO 40
H1 = S1
S1 = S2
S2 = H1
H1 = H8
H8 = K8
K8 = H1
C-----
C LIMITING RESULTS FOR DMIN1(S1,S2) = 0
C-----

```

```

40 SEPS1 = DMIN1(6.71D0*SQEPS, 1.D-5)
C-----  

C      R = K, S2 SMALL  

C-----  

C      IF (K8 .NE. R) GOTO 45
C      YY = .166484D0*(R*(H8*H8 + 1.0D0) + 1.0D0/R)*S2
C      IF (DABS(YY) .GT. SEPS1) GOTO 85
C      H1 = H8/RT2
C      P = C*DEXP(-H1*H1)*DSQRT(K8*S2)
C      RETURN
C-----  

C      R .GT. K, S2 SMALL  

C-----  

45 IF (R .LT. K8) GOTO 85
IF (K8 .NE. 0.0D0) GOTO 75
H1 = S2*S2/(4.0D0*RT2*R)
G2 = G2/RT2
IF (J .EQ. 0) G2 = DABS(H8 - R)/RT2
IF (DABS(G2) .LT. 4.0D0) GOTO 55
50 IF (H1*DABS(G2) .GT. SEPS1) GOTO 85
P = .5D0*AERF(H8/RT2,R/RT2)
RETURN
55 IF (J .NE. 0) GOTO 60
P = .5D0*AERF(H8/RT2,R/RT2)
GOTO 70
60 IF (H8 + R .LT. RT2) GOTO 65
P = .5D0*(ERFC(G2) - ERFC((H8 + R)/RT2))
GOTO 70
65 P = .5D0*(ERF((H8 + R)/RT2) - ERF(G2))
70 IF (H1*DEXP(-G2*G2) .GT. P*SEPS1) GOTO 85
RETURN
75 Z = (R - K8)*(R + K8)
G2 = DSQRT(Z)
H1 = DABS(H8 - G2)
J = 0
IF (H1 .GT. 5.D0) GOTO 80
J = 1
Z8 = AERF(H8/RT2,G2/RT2)
IF (Z8 .EQ. 0.0D0) GOTO 85
H1 = DEXP(-0.5D0*(H8 - G2)**2)/Z8
80 H1 = 1.0D0/(8.0D0*RT2*Z)*(K8*K8*DABS(H8 - G2) + R*R/G2)*S2*S2*H1
IF (DABS(H1) .GT. SEPS1) GOTO 85
U = .5D0
IF (J .EQ. 0) Z8 = AERF(H8/RT2,G2/RT2)
S2 = RT2*S2
IF (K8 - R .GT. -13.D0*S2) U = .25D0*AERF(K8/S2,R/S2)
P = U*Z8
RETURN

```

```

C-----
C      FIND THE VALUE OF A
C-----
85  J8 = 0
    S0 = S1*S1
    S9 = S2*S2
    G3 = H8*H8
    G2 = K8*K8
    Z8 = S0 + S9
    Z = S0*S0 + S9*S9
    H3 = G3*S0 + G2*S9
    T1 = 2.0D0*(Z + 2*H3)
    YY = R*R*(H2 + Z8)/T1
    T1 = (H2 + Z8)*(H2 + Z8)/T1
    CALL GRATIO(T1,YY,Z,Z8,0)
    Z2 = R/(RT2*S2)
    R8 = R/(RT2*S1)
    H2 = H8/(RT2*S1)
    H3 = K8/(RT2*S2)
    S0 = 0.D0
    S9 = 0.D0
    IF (Z .LT. 1.D-13 .AND. S2 .GT. 5.D-10) GOTO 90
    IF (H2 .GT. 50.0D0 .OR. H3 .GT. 50.0D0) S0 = 1.5D0
90  U = AERF(H3,Z2)
    YY = .25D0*U*AERF(H2,R8)
    IF (YY .GT. 0.1D0) S9 = .5D0
    IF (YY .GE. 5.D-15) GOTO 95
    Z = YY
    GOTO 100
95  Z = DMIN1(YY,Z)
100 IF (Z .GE. 5.D-1) GOTO 105
    A = A + .5D0
    IF (Z .GE. 5.D-4) GOTO 105
    A = A + .5D0 + S0
    IF (Z .GE. 1.D-6) GOTO 105
    A = A + .5D0 + S9
    IF (Z .GE. 5.D-9) GOTO 105
    A = A + .5D0
    IF (Z .GE. 5.D-10) GOTO 105
    A = A + .25D0
    IF (Z .GE. 5.D-11) GOTO 105
    A = A + .25D0
    IF (Z .GE. 5.D-12) GOTO 105
    A = A + .25D0
    IF (Z .GE. 5.D-13) GOTO 105
    A = A + .5D0 + S9/2
    IF (Z .GT. 5.D-15) GOTO 105

```

```

A = A + .50D0
IF (Z .GT. 1.D-18) GOTO 105
A = A + .25D0
IF (Z .GT. 1.D-20) GOTO 105
A = A + .25D0
IF (Z .GT. 1.D-25) GOTO 105
A = A + 1.D0
IF (Z .GT. 1.D-30) GOTO 105
A = A + 2.D0
105 IF (S2 .GE. 5D-2 .OR. H3 .LE. Z2) GOTO 107
T9 = R8*U*DEXP(-H2*H2)
IF (T9 .LT. 5D-2*Z) A = A + .5D0
C-----
C      START INTEGRATION PROCEDURE
C-----
107 S0 = S1
S9 = S2
Z8 = S2
G2 = K8
G3 = H8
Z9 = S1
C-----
C      DETERMINE INTERVAL OF INTEGRATION, (A,B).
C      E3 = (B-A)/2,   D1 = (B+A)/2.
C-----
110 Z = G2 + A*Z8
H3 = G2 - A*Z8
H5 = 0.0D0
T3 = -1.0D0
A1 = (G3 - A*Z9)/R
IF (DABS(A1 - 0.5D0) .GE. 0.5D0) GOTO 115
A2 = ((1.0D0 - G3/R) + A*Z9/R)*(1.0D0 + A1)
SA = DSQRT(A2)
IF (SA .LT. H3/R) GOTO 115
T3 = A1/DSQRT(1.0D0 + SA)
C-----
C      T9 = 1.0 - D1
C-----
115 IF (H3 .GT. 0.0D0) GOTO 135
C-----
C      H3 .LE. 0.0
C-----
IF (T3 .LT. 0.0D0) GOTO 120
D2 = 0.5D0*(1.0D0 + T3)
E4 = 0.25D0*SA/D2
T5 = E4
120 IF (Z .LT. R) GOTO 125

```

```

E3 = .5D0
D1 = E3
T9 = D1
GO TO 130
125 D1 = .5D0*(1.0D0 + DSQRT(1.0D0 - Z/R))
E3 = .25D0*Z/(R*D1)
T9 = E3
130 IF (T3 .LE. D1 - E3) GOTO 165
GOTO 160
C-----
C      H3 .GT. 0.0
C-----
135 H5 = 1.0D0
Q = H3/R
IF (Q .LE. 1.0D0) GOTO 140
E3 = .5D0
D1 = E3
T9 = D1
GOTO 165
140 IF (T3 .LT. 0.0D0) GOTO 145
E4 = DSQRT((1.0D0 - G2/R) + A*Z8/R)
D2 = 0.5D0*(E4 + T3)
T5 = 0.5D0*(Q/(1.0D0 + E4) + SA/(1.0D0 + T3))
E4 = 0.25D0*(SA - Q)/D2
145 IF (Z .LT. R) GOTO 150
E3 = 0.5D0*DSQRT((1.0D0 - G2/R) + A*Z8/R)
D1 = E3
T9 = 0.25D0*(3.0D0 + H3/R)/(1.0D0 + E3)
GOTO 155
150 T9 = DSQRT((1.0D0 - G2/R) + A*Z8/R)
T2 = DSQRT(1.0D0 - Z/R)
D1 = 0.5D0*(T9 + T2)
E3 = A*Z8/(R*2.0D0*D1)
T9 = 0.5D0*((H3/R)/(1.0D0 + T9) + (Z/R)/(1.0D0 + T2))
155 IF (T3 .LE. D1 - E3) GOTO 165
160 E3 = E4
D1 = D2
T9 = T5
165 IF (J8 .NE. 0) GOTO 170
J8 = 1
F = E3
T = D1
T1 = T9
H6 = H5
Z8 = S1
G2 = H8
G3 = K8

```

```

Z9 = S2
GOTO 110
C-----
C  DETERMINE IN WHICH ORDER THE X AND Y INTEGRATIONS ARE CARRIED OUT.
C-----
170 IF (S2 .GT. 2D-2 .AND. H8 + K8 .LT. 2D2) GOTO 172
    IF (DABS(E3 - F) .GT. .4D0*F) GOTO 172
    IF (D1 - T) 200, 180, 195
172 IF (E3 .LT. 2.D4*EPS) GOTO 195
    IF (F .LT. 2.D4*EPS) GOTO 200
    IF (DMAX1(H8/S1,K8/S2) .LT. 2.0D0) GOTO 175
    IF (S2 .LT. 1.D-5) GOTO 175
    IF (S2 .GE. 5.D-4) GOTO 180
    IF (S1 .NE. S2) GOTO 195
175 IF (DMIN1(E3, F) .LT. 2.5D-2*SQEPS) GOTO 185
180 IF (E3 - F) 200, 190, 195
185 IF (E3 - F) 195, 190, 200
190 IF (S0 .GE. S9) GOTO 200
195 E3 = F
    D1 = T
    T9 = T1
    S9 = S1
    S0 = S2
    Z8 = H8
    H8 = K8
    K8 = Z8
    H5 = H6
200 Z2 = R/(RT2*S9)
    R8 = R/(RT2*S0)
    H2 = H8/(RT2*S0)
    H3 = K8/(RT2*S9)
    N1 = 16
203 IZ = 0
    IZ3 = 0
    IY = 0
    P = 0.D0
    T = H2 - R8 + R8*(D1 - E3*V(16))**2
    IF (T .GT. 0.D0 .AND. T*T .GT. -DXPAR(1)) RETURN
    Q1 = RPI*E3*R8
    G3 = 0.0D0
    NT = 2*N1 + 1
    Z = (.5D0 + D1) +.5D0
    DO 260 II = 1, NT
205      I = II - (N1 + 1)
        IF (I .EQ. 0) GOTO 260
        J = IABS(I)
        Q = E3*ISIGN(1,I)*V(J)

```

```

T = Q + D1
H6 = Z + Q
Q = T9 - Q
F = H6*Q
T1 = R8*F
T2 = (H2 - T1)*(H2 - T1)
IF (H2 - R8 .GE. 0.D0 .OR. T .LT. EPS) T2 =
*   ((H2 - R8) + R8*T*T)**2
T4 = DEXP(-T2)
IF (H2 .NE. 0.0D0) GOTO 210
T4 = T4 + T4
GO TO 215
210 IF (H5 .NE. 0.0D0) GOTO 215
T2 = 4.0D0*H2*T1
IF (T2 .GT. C8) GO TO 215
T4 = T4*(1.0D0 + DEXP(-T2))
215 IF (IZ .NE. 0) GOTO 255
G2 = DSQRT(1.0D0 + F)
Z1 = Z2*T*G2
X1 = H3 - Z1
IF (X1 .GT. -A) GOTO 225
220 IZ = 1
X5 = 2.D0
GO TO 255
225 IF (DABS(X1) .GT. 1.D-2*Z1) GOTO 230
IY = 1
X1 = ((K8 - R) + R*F*F/(1.D0 + T*G2))/(RT2*S9)
230 IF (IZ3 .NE. 0) GOTO 250
SA = H3 + Z1
IF (SA .GT. A3) GOTO 245
IF (IY .EQ. 0) GOTO 240
IF (X1 .GT. RT2) GOTO 235
X5 = ERF(SA) - ERF(X1)
GOTO 255
235 X5 = ERFC(X1) - ERFC(SA)
GOTO 255
240 X5 = AERF(H3,Z1)
GOTO 255
245 IZ3 = 1
250 X5 = ERFC(X1)
255 G3 = G3 + X5*T4*T*W(J)
260 CONTINUE
P = Q1*G3
IF (P .GT. YY) P = YY
IF (P .GT. (1.0D0 - DMIN1(1.D6*EPS, 1.D-5))) P = 1.0D0
IF (P .LT. 0.0D0) P = 0.0D0
RETURN
END

```

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